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LETTER TO THE EDITOR

Normal forms for generalised Hopf bifurcations

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Abstract. Explicit coefficients of normal forms for generalised Hopf bifurcations are calculated to the seventh (but, to save space, they are given only to the fifth) order near the critical point. The real part of the third-order coefficient reproduces, at the critical point, the third-order focal value of Andronov *et al.*

Of recent interest in non-linear dynamics is the study of dramatic changes in the behaviour of non-linear self-organising systems. One method, among others, of analysing these systems has now become systematic and standard: this can be outlined as follows [1]. The systems are usually described by a set of ordinary (or partial) differential equations with a large (even infinite) number of variables. Near the critical points of linear stability analysis, reduction procedures, such as centre manifold theory [2] or, more generally, the slaving principle [3, 4], allow one to reduce the number of equations considerably. The resulting low-dimensional set of equations, known as generalised Ginzburg–Landau equations (GGLE), govern the evolution in or near the centre manifold. The GGLE are usually of quite general form. Normal form theory, originated by Poincaré [5] and Birkhoff [6] and recently advanced by Takens [7], Bogdanov (see [5, ch 6]) and others, can be used to put the GGLE of general form into normal form, which is different for different kinds of bifurcations.

Normal form theory has been developed by mathematicians. Concrete calculations of the coefficients of each normal form are of great interest to physicists and to those researchers in applied fields. One such attempt has recently been made by Knobloch [8] who calculated the normal form for the double-zero bifurcations. Another interesting bifurcation is the Hopf bifurcation or generalised Hopf bifurcation, i.e. a Hopf bifurcation with higher codimension, for which Golubitsky and Langford [9] have given a complete and systematic classification and all unfoldings for the cases where the second (H2) or third (H3) or both of the Hopf hypotheses fail, but with the restriction that the codimension is less than three. In this letter, we give the results of our calculations of the coefficients of the normal form for the case where only hypotheses H1 and H2 (see [9]) are satisfied.

Near the Hopf bifurcation point, the slaving principle leads to a complex GGLE of the form [1]

$$\mathrm{d}u/\mathrm{d}t = \lambda_u u + P_2(u, u^*) + \ldots + P_n(u, u^*) + \ldots \tag{1}$$

where u^* stands for the complex conjugate of u and P_n is defined as

$$P_n(u, u^*) = C_{n0}u^n + C_{n-11}u^{n-1}u^* + \ldots + C_{0n}u^{*n}. \tag{2}$$

Equation (1) can be transformed into a normal form

$$\mathrm{d}z/\mathrm{d}t = (\lambda_u + g_3|z|^2 + g_5|z|^4 + g_7|z|^6 + \ldots)z \tag{3}$$

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by successive non-linear transformations. The RHS of (3), generally an infinite series, contains only the terms which are resonant at the exact critical point $\mathrm{Re}\,\lambda_u = 0$. How many terms should be retained depends on how high the codimension of the RHS of (3) is. The original Hopf bifurcation is a codimension-1 bifurcation with $g_3 = 0$, so that g_5 and higher-order terms can be safely neglected. The case $g_3 = 0$ but $g_5 \neq 0$ is a codimension-2 Hopf bifurcation. In this case, the calculation of g_5 is necessary. If we have more parameters to vary so that $g_3 = g_5 = 0$ we have a codimension-3 Hopf bifurcation and the calculation of g_7 is required. All higher-order terms than g_7 can be omitted. The transformation from (1) to (3) is given by

$$u = z + Q_2(z, z^*) + Q_3(z, z^*) + \ldots \tag{4}$$

with

$$Q_n = B_{n0}z^n + B_{n-11}z^{n-1}z^* + \ldots + B_{0n}z^{*n}. \tag{5}$$

The central problem is to determine the transformation (4) and to give the explicit expressions of g_i in terms of C_{ij}. The solutions of (3) were studied by Takens [7] and the bifurcation diagram by Golubitsky and Langford [9], but the explicit coefficients are unknown. The result of Andronov *et al* [10] and Marsden and McCracken [11] are equivalent to $\mathrm{Re}(g_3)$ at the critical point. In order to cope with high codimension problems and to calculate the oscillatory motion near the critical point, we have to generalise the previous results in two directions: (i) to calculate higher-order coefficients of the normal form than g_3 and (ii) to release the restriction to exact critical points.

An essential point in constructing the transformation near the critical points is the following observation. According to the theory of Poincaré [5], one can, in principle, transform away all non-linearities in equation (1) because the eigenvalues are no longer resonant away from the critical point. But the transformation so determined contains some singularities at the exact critical point. In the applications, however, we usually intend to pass through the critical points smoothly with the variation of control parameters. Therefore we must retain terms such as $|z|^{2n}z$, which are resonant at the critical point. Correspondingly, we let $B_{nn-1} = 0$ in the transformation (4).

Using the above ideas, we have made a calculation of g_3, g_5 and g_7 using the symbolic manipulation program SMP. Due to the length of the expression for g_7, we give only the results for g_3 and g_5:

$$B_{20} = C_{20}/\lambda_u$$

$$B_{11} = C_{11}/\lambda_u^*$$

$$B_{02} = C_{02}/(2\lambda_u^* - \lambda_u)$$

$$B_{30} = (C_{30} + 2B_{20}C_{20} + B_{02}^*C_{11})/2\lambda_u$$

$$g_3 = C_{21} + 2B_{11}C_{20} + (B_{11}^* + B_{20})C_{11} + 2B_{02}^*C_{02}$$

$$B_{12} = [C_{12} + 2B_{02}C_{20} + (B_{11} + B_{20}^*)C_{11} + 2B_{11}^*C_{02}]/2\lambda_u^*$$

$$B_{03} = (C_{03} + B_{02}C_{11} + 2B_{20}^*C_{02})/(3\lambda_u^* - \lambda_u)$$

$$B_{40} = [C_{40} + 3B_{20}C_{30} + B_{02}^*C_{21} + (2B_{30} + B_{20}^2)C_{20} + (B_{02}^*B_{20} + B_{03}^*)C_{11} + B_{02}^{*2}C_{02}]/3\lambda_u$$

$$\begin{aligned} B_{31} = [&C_{31} - 2g_3B_{20} + 3B_{11}C_{30} + (B_{11}^* + 2B_{20})C_{21} + 2B_{02}^*C_{12} + 2B_{11}B_{20}C_{20} \\ &+ (B_{12}^* + B_{11}^*B_{20} + B_{30} + B_{02}^*B_{11})C_{11} + 2(B_{02}^*B_{11}^* + B_{03}^*)C_{02}]/(2\lambda_u + \lambda_u^*) \end{aligned}$$

$$B_{22}=[C_{22}-(g_3+g_3^*)B_{11}+3B_{02}C_{30}+3B_{02}^*C_{03}+(2B_{11}+B_{20}^*)C_{21}+(2B_{11}^*+B_{20})C_{12}$$
$$+(2B_{12}+B_{11}^2+2B_{02}B_{20})C_{20}+(B_{11}B_{11}^*+B_{20}B_{20}^*+B_{02}B_{02}^*)C_{11}$$
$$+(B_{11}^{*2}+2B_{12}^*+2B_{02}^*B_{20}^*)C_{02}]/(2\lambda_u^*+\lambda_u)$$
$$B_{13}=[C_{13}-2g_3^*B_{02}+2B_{02}C_{21}+(B_{11}+2B_{20}^*)C_{12}+3B_{11}^*C_{03}+2(B_{02}B_{11}+B_{03})C_{20}$$
$$+(B_{30}^*+B_{12}+B_{02}B_{11}^*+B_{11}B_{20}^*)C_{11}+2B_{11}^*B_{20}^*C_{02}]/3\lambda_u^*$$
$$B_{04}=[C_{04}+B_{02}C_{12}+3B_{20}^*C_{03}+B_{02}^2C_{20}+(B_{03}+B_{02}B_{20}^*)C_{11}$$
$$+(2B_{30}^*+B_{20}^{*2})C_{02}]/(4\lambda_u^*-\lambda_u)$$
$$g_5=C_{32}+4B_{02}C_{40}+(3B_{11}+B_{20}^*)C_{31}+2(B_{11}^*+B_{20})C_{22}+3B_{02}^*C_{13}+3B_{03}^*C_{03}$$
$$+3(B_{12}+B_{11}^2+2B_{02}B_{20})C_{30}$$
$$+(2B_{02}B_{02}^*+2B_{11}B_{11}^*+2B_{11}B_{20}+2B_{20}B_{20}^*)C_{21}$$
$$+(2B_{12}^*+B_{30}+B_{11}^{*2}+2B_{02}^*B_{11}+2B_{02}^*B_{20}^*+2B_{11}^*B_{20})C_{12}+6B_{02}^*B_{11}^*C_{03}$$
$$+2(B_{22}+B_{02}B_{30}+B_{12}B_{20})C_{20}+(B_{22}^*+B_{31}+B_{02}B_{03}^*+B_{02}^*B_{12}+B_{11}B_{12}^*$$
$$+B_{20}^*B_{30})C_{11}+2(B_{03}^*B_{20}^*+B_{11}^*B_{12}^*+B_{13}^*)C_{02}.$$

With the above expressions, we can easily obtain the amplitude and renormalised frequency of the oscillatory motion near the critical point, not only for soft (second-order transition, codimension 1) but also for hard (first-order transition, higher codimention) onset of oscillation. In particular, g_3 and g_5 can be used to judge analytically whether a codimension-2 or 3 Hopf bifurcation is possible or not. At the exact critical point, the real part of g_3 reduces to

$$\mathrm{Re}(g_3)_c=\mathrm{Re}(C_{21})-\mathrm{Im}(C_{11}C_{20})/\mathrm{Im}(\lambda_u)$$

which is in complete accord with the third-order focal value of Andronov *et al* [10], if we write it in terms of real coefficients. But it is simpler and more concise than that given there, due to the use of complex variables. General connections between normal form theory and Andronov's focal values are now being made. The corresponding results will be given elsewhere.

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